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A Use of Spline Data Techniques with Optimization Programs

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OVER the past few years a great deal of interest and effort has been directed towards the optimization of trajectories. However, in using some of the optimization schemes numerical difficulties have been encountered which, at times, seem to be problem dependent; problems involving only a small portion of flight time in the atmosphere have generally been successfully solved, whereas problems entailing increased portions of atmospheric flight time have tended to have less likelihood of success. One reason for the additional difficulties in atmospheric flight is believed to stem from the way in which aerodynamic data curves are usually input to the problem. This Note discusses one way in which errors are introduced into such problems and offers one technique for the elimination of such errors.

A widely used method of curve representation consists of simple tabular input data with linear interpolation used for evaluation between the data points. For example, in the aircraft flight differential equations of Eq. (1), input data are used to represent thrust, T, mass flow rate, $\dot{m}$, and the lift and drag coefficients (C_L and C_D, respectively);

$$\dot{v} = (T/m)\cos\alpha - \rho(V^2S/2m)C_D - g\sin\gamma$$
$$\dot{\gamma} = \rho(VS/2m)C_L + (T/mV)\sin\alpha - g\cos\gamma/V \qquad (1)$$
$$\dot{h} = V\sin\gamma,\ \dot{x} = V\cos\gamma,\ \dot{m} = \dot{m}(h,M)$$

or, in general vector form

$$\dot{x} = f(x,\alpha,t)$$

However, in using the simple tabular input form of data curve representation, it is important to realize that, at the data points, the slope of the represented curve is discontinuous. When an optimization problem is attempted using either a direct or indirect method, these discontinuities play an important role as, with both methods, numerical integration of an auxiliary set of differential equations of the vector form

$$\dot{\lambda} = -f_x{}^T\lambda \qquad (2)$$

where

$$\dot{x} = f(x,\alpha,t)$$

is required. In both methods, the discontinuities of the slope of the input data curves are reflected as jump discontinuities on the right side of Eq. (2) in the auxiliary variables. Recognition of this error source makes it desirable to find a means of data curve representation that exhibits continuous first derivatives.

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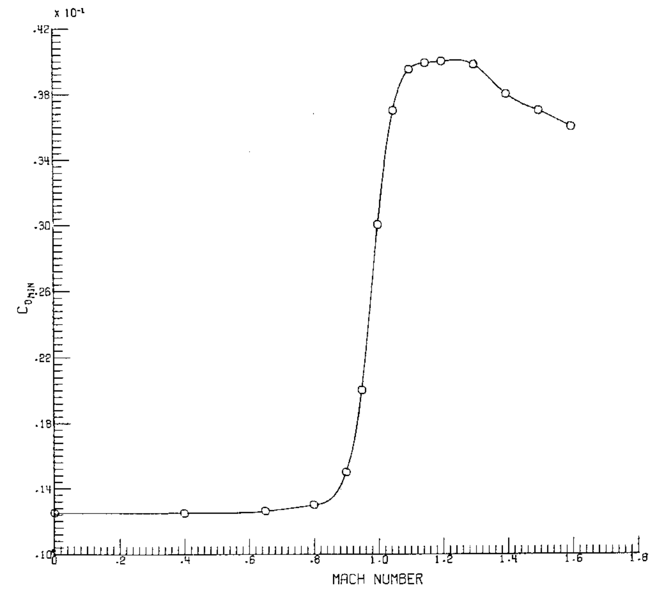

Fig. 1 Example of spline curve fit to a set of input points.

Several methods of curve fittings are now in general use. Recently the spline function has been generating considerable interest. See for example Refs. 1–3. In particular the cubic spline function appears to be quite useful since in certain applications the cubic spline minimizes the integral of the curvature over the entire function.[4] By construction each segment of the function, i.e., between input data points, is a cubic polynominal, such that adjoining cubics and their first two derivatives are continuous. Thus the cubic spline representation of input data curves represent one way of eliminating the difficulties resulting from the discontinuous first derivatives.

To evaluate the benefits possible through the use of such a data representation, a simple minimum time-to-climb aircraft flight-path problem was set up. The aircraft model chosen was one illustrative of a supersonic interceptor. In representing the input data curves, tabular data points were picked off the plots so as to provide for a reasonable fit to the data curves. As an example of the representation provided by the tabular input data format, Fig. 1 shows a choice of data points which might be used to represent the minimum drag coefficient, $C_{D_{\min}}$, as a function of Mach number. In this case, the input data points are shown as the symboled points, and straight line interpolation is then used between the points. Such a representation is normally in use with most optimization programs. However, using the same data points, a cubic spline fit may be made through the points using a spline routine similar to the one described in Ref. 4. Such a fit is shown as the solid line of Fig. 1 and may be seen to have a continuous first derivative. Cubic spline fits were also made to the thrust and mass flow data and other aerodynamic data in preparation for use with a digitaloptimiization program.

The problem solution was attempted with two different computer programs employing different optimization schemes. One of the programs used a calculus of variations approach to solve the problem. In using this method, one of the necessary conditions for a calculated trajectory to be optimal is that the Hamiltonian of the system, computed by the matrix equation

$$H = p^T\dot{x} - 1 \qquad (3)$$

where

$$\dot{p} = -f_x{}^Tp$$

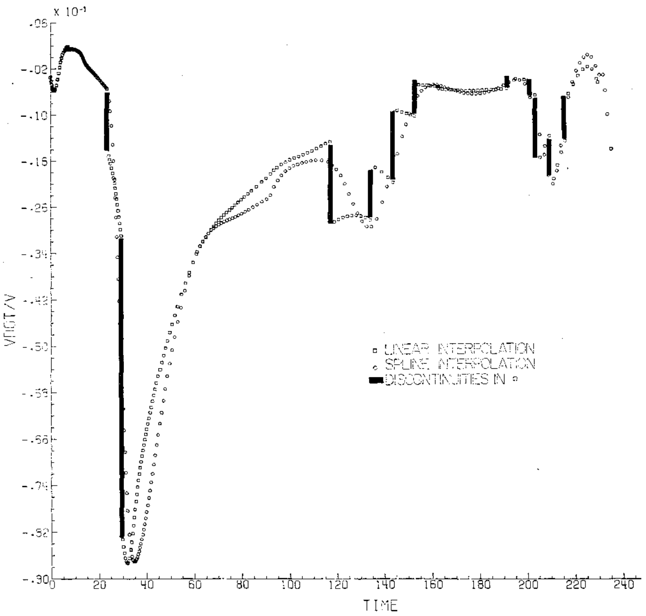

Fig. 2 Comparison of partial derivative time history using linear and spline interpolation.

be constant. When using the linear interpolation scheme for obtaining input data values, the Hamiltonian, which is supposed to remain constant over the entire trajectory, had essentially step changes occuring at various points throughout the trajectory. Examination of the f_x matrix showed that the jumps in the value of the Hamiltonian resulted from changes in the first derivative of the input data. Equation (4) shows, by way of example, the vector of the f_x matrix associated with the partial derivatives taken with respect to velocity;

$$f_v = \begin{Bmatrix} \dfrac{\partial \dot{v}}{\partial v} \\ \dfrac{\partial \dot{\gamma}}{\partial v} \\ \dfrac{\partial \dot{h}}{\partial v} \\ \dfrac{\partial \dot{x}}{\partial v} \\ \dfrac{\partial \dot{m}}{\partial v} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\cos\alpha}{ma}\dfrac{\partial T}{\partial M} - \dfrac{2D}{mV} - \dfrac{D}{ma}\dfrac{1}{C_D}\dfrac{\partial C_D}{\partial M} \\ g\dfrac{\cos\gamma}{V^2} + \dfrac{L}{mV^2} + \dfrac{L}{mVa}\dfrac{1}{C_L}\dfrac{\partial C_L}{\partial M} - \dfrac{T\sin\alpha}{mV^2} + \dfrac{\sin\alpha}{mVa}\dfrac{\partial T}{\partial M} \\ \sin\gamma \\ \cos\gamma \\ \dfrac{1}{a}\dfrac{\partial \dot{m}}{\partial M} \end{Bmatrix} \quad (4)$$

However, in using the cubic spline method to evaluate input data curves, the Hamiltonian of the system does remain constant to at least eight places on the digital computer, thereby satisfying a necessary condition for a true optimal trajectory.

The same problem solution was also attempted with a computer program using the method of steepest descent.[5] In this method the discontinuities appear in the various terms of the adjoint equation integrand, as for example in the adjoint equations associated with velocity;

$$\dot{\lambda}_v = -f^T\lambda$$

$$\dot{\lambda}_v = -(\partial\dot{v}/\partial v)\lambda_v - (\partial\dot{\gamma}/\partial v)\lambda_\gamma - (\partial\dot{h}/\partial v)\lambda_h - (\partial\dot{x}/\partial v)\lambda_x - (\partial\dot{m}/\partial v)\lambda_m \quad (5)$$

Since the adjoint variables are used in determining the perturbation in the control history, the lack of a continuous set of first derivatives once again introduces an error. As an example of the errors introduced, the adjoint term $\partial\dot{v}/\partial v$, shown in Eq. (5), is shown as a function of trajectory time in Fig. 2 for both types of data representation. The value of $\partial\dot{v}/\partial v$ using linear interpolation is shown by the square symbol points on this figure and may be seen to be discontinuous at certain time points which may be shown to correspond to conditions at input data points. This discontinuous behavior is detrimental, of course, because the adjoint values are used in determining the change in the control necessary to improve the trajectory. Also as a result of this, discontinuities will appear in the control history. For comparison, the same term, $\partial\dot{v}/\partial v$, is shown when the spline representation is used as indicated by the circles. Note that in this case, a continuous curve is described. Correspondingly, the spline data representation then provides for a continuous control history.

Thus the use of the spline data representation when used with both of the optimization methods was shown to offer an improvement over the linear interpolation scheme. The steepest descent program exhibited more consistency in steadily decreasing the payoff gradient over the successive iterations, and the program logic was better able to predict the changes in payoff and performance which could be obtained. The program employing the calculus of variations method was able to compute trajectories over which the Hamiltonian of the system remained constant, thereby satisfying a necessary condition for an optimal trajectory. These improvements in program operation are the result of removing the slope discontinuities in the thrust, mass flow, and aerodynamic data curve representations. This slope difficulty is particularly objectionable with the atmospheric flight problems because of the extensive use of aerodynamic data, but similar slope difficulties also result from slope changes in the mass flow and thrust tabular data representations. In conclusion then, with the given sample problem, the optimization programs appeared to behave better when using the spline data representation due to the elimination of an error source in the problem setup.

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